Name:________________________________

Lessons 3 and 4
Integration by Substitution
Math 16020

1 Integration Rule Resulting from the Chain Rule

In the last class, we frequently used the sum and difference rules when integrating.

(a) $\displaystyle\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

(b) $\displaystyle\int (f(x) - g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$

Unfortunately, there are no simple integration rules for ______________________ and ______________________.

Example 1. Evaluate

(a) $\displaystyle\int (x^2)(x^3)\, dx$

(b) $\displaystyle\int (x^2)\, dx \int (x^3)\, dx$

(c) What are you supposed to learn from this example?

Although there is not a simple integration formula for products, every derivative rule is a associated with an integral rule. The Chain Rule for derivatives does lead to an integration formula that will help us to integrate some special products.

Theorem 2 (The Chain Rule)**.** If g is differentiable at x and f is differentiable at $g(x)$, then

$$[f(g(x))]' = f'(g(x))g'(x)$$

This derivative rule leads to the integration formula

$$\rule{8cm}{0.4pt}$$

If we make the substitution

$$u = g(x),$$

then

$$\frac{du}{dx} = \rule{4cm}{0.4pt} \quad \Longrightarrow \quad du = \rule{4cm}{0.4pt}$$

so we can rewrite our integration formula as

$$\int f'(g(x))g'(x)\,dx = \rule{8cm}{0.4pt}$$

Example 3. $\displaystyle\int (3x^2)(x^3+10)^{15}\,dx$

2 Substitution Examples

As we do these examples, here are some things to keep in mind.

1. Substitution might be a good tool to try if you are working with a product or a quotient (which is a product in disguise).
2. If you are going to use substitution, the function you are trying to integrate *often* contains a function **and its derivative** - *up to a constant multiple.*
3. The function you choose for u is *often* inside of another function.
4. If you set $u = g(x)$ so that $du = g'(x)dx$, then ALL of $g'(x)$ is multiplied by dx. For example, if $u = x^2 + 3x$, then
$$du = (2x + 3)dx$$
It is **NOT** correct to write

   ____

5. After you find du, you can multiply or divide both sides of your equation **by a constant** to make the substitution easier.
6. After you make the substitution, all of the variables in your new integral should be u's. Your integral should **ONLY CONTAIN ONE VARIABLE** - and it should be easier to solve.
7. When you are finding an indefinite integral (no limits) using substitution, your last step is **BACK SUBSTITUTION**. Your **final answer** should be in the **ORIGINAL VARIABLE**.

Example 4. $\int 3x\sqrt{x^2+5}\,dx$

Example 5. $\int (3x+4)^{17}\,dx$

3 Substitution with Definite Integrals

Whenever you are using substitution to solve a definite integral (with limits), you must **USE YOUR SUBSTITUTION EQUATION** to **CHANGE THE LIMITS TO u-LIMITS**.

Example 6. $\int_0^1 2x^6 e^{x^7+3}\, dx$

Example 7. $\int_0^{\frac{\pi}{3}} \sin(x)\cos^4(x)\, dx$

4 More examples

Example 8. A function $f(x)$ has tangent line slope $x\sqrt{x-2}$ for all $x > 2$. The graph of f passes through the point $\left(3, \frac{9}{15}\right)$. Find a formula for $f(x)$.

Example 9. Find the area under the curve $y = 3\sin(0.5x)$ from $x = 0$ to $x = \pi$.

Example 10. $\displaystyle\int_0^{12} \frac{x}{\sqrt{x+4}}\,dx$

Example 11. The area under the curve $3e^{0.2x}$ on the interval $0 \leq x \leq a$ is 45. What is a?